MA 3046 - Matrix Analysis
Perspectives on Systems of Linear Equations

Virtually all discussion of any form of equations, including systems of linear equations

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & a_{2n}x_n & = & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & a_{mn}x_n & = & b_m
\end{array}
$$

or

$$\mathbf{A}\ \mathbf{x} \ = \ \mathbf{b}$$

revolves about the three fundamental questions of:

- *Existence* – are there **any** solutions?
- *Uniqueness* – if so, **how many** are there?
- *Construction* – how do you find them?

At least for our purposes at this point in this course, we can consider the third question to have been already answered – Gaussian Elimination. (Although we would caution you that, at a more sophisticated level, when one considers issues involved in solving "real" applications on real computer systems, non-trivial issues arise concerning both *efficiency* – how much core and CPU time are required to execute the algorithm – and *accuracy* – how many significant digits can be expected in the resulting computed solution.)

One of the very attractive features of systems of linear equations is that twin issues of existence and uniqueness may be looked at from a number of very different, yet all equally valid perspectives. These are

- *Gaussian Elimination* – a purely computational viewpoint, which focuses on (elementary) row operations which reduce the augmented matrix

  $$[\ \mathbf{A}\ \ \mathbf{b}\ ]$$

  to row-echelon form, and answer the questions of existence and uniqueness, respectively, based on whether a pivot exists in each row, and whether or not there are any free columns.
- *Transformation* – a pictorial viewpoint model, which matrix multiplication as a simple, input/output process

  $$\mathbf{x} \quad \longrightarrow \quad \boxed{} \quad \longrightarrow \quad \mathbf{A}\ \mathbf{x}$$

  and treats existence as question of whether or not a specific output is feasible (i.e. in the range of the transformation), and uniqueness as one of whether or not there are inputs (in the domain) which produce a zero (i.e. not measurable) output.

- *Input Orientation* – a row-oriented model which views the system as describing a simultaneous intersection of geometric entities (lines, planes, etc.). From this perspective, each equation (row) defines a distinct geometric entity in the coordinate space defined by the unknowns, existence reduces to the question of whether any intersections are possible, and uniqueness becomes the question of whether the entities can intersect in something more than a single point.
- *Output Orientation* – a column-oriented model which views the system as a involving linear combinations of the columns of the matrix **A**. From this perspective, the unknowns are interpreted as the weights applied to the various column vectors, existence reduces to the study of the span of the columns, and uniqueness becomes the question of linear independence of these columns.

These different perspectives can be illustrated by considering the system:

$$\begin{array}{rcrcr} 3x_1 & + & x_2 & = & 1 \\ x_1 & - & x_2 & = & 2 \end{array}$$

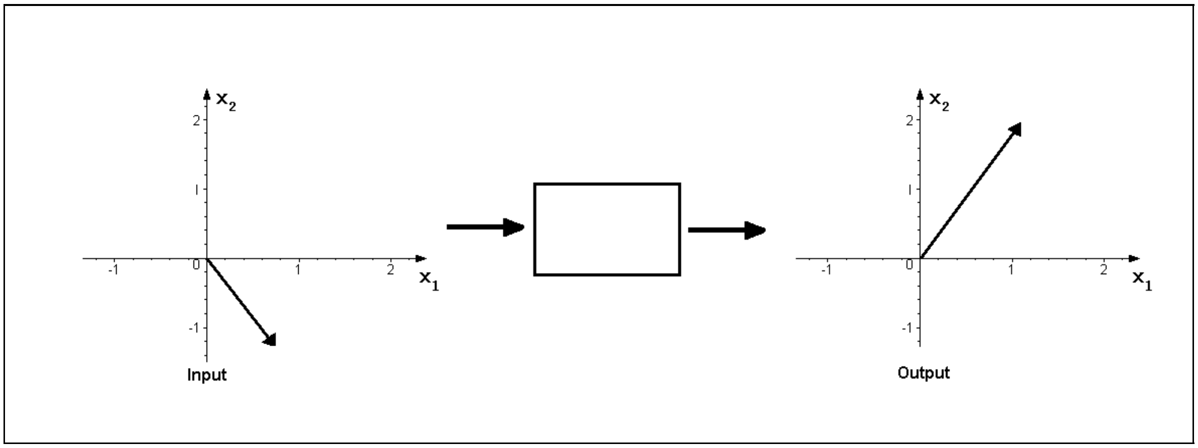
From the perspective of Gaussian elimination, this problem simply involves forming the augmented matrix

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \Longrightarrow \begin{cases} x_1 = \frac{3}{4} \\ x_2 = -\frac{5}{4} \end{cases}$$

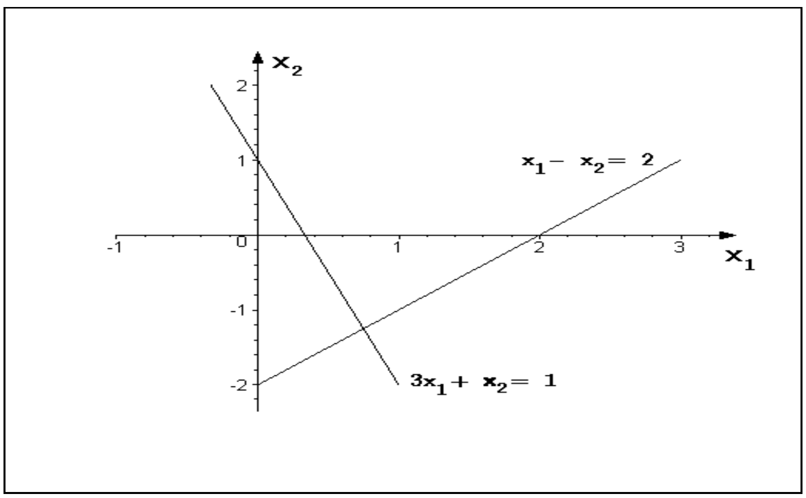
From the transformation perspective, this problem simply involves the fact that

$$\underbrace{\begin{bmatrix} \frac{3}{4} \\ -\frac{5}{4} \end{bmatrix}}_{\text{Input}} \longrightarrow \boxed{} \longrightarrow \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ -\frac{5}{4} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{Output}}$$

or, pictorially, showing that



From the input model perspective, the problem involves finding the intersection of the two straight lines shown below



where x_1 and x_2 are, respectively, the coordinates of the intersection, Lastly, from the output model perspective, the problem considers what weights are required to construct the vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{as a linear combination of} \qquad \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

As shown in the figure below, these weights are $\frac{3}{4}$ and $-\frac{5}{4}$, respectively.